Superselection rules induced by infrared divergence

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Abstract

Superselection rules induced by the interaction with a mass zero Boson field are investigated for a class of exactly soluble Hamiltonian models. The calculations apply as well to discrete as to continuous superselection rules. The initial state (reference state) of the Boson field is either a normal state or a KMS state. The superselection sectors emerge if and only if the Boson field is infrared divergent, i. e. the bare photon number diverges and the ground state of the Boson field disappears in the continuum. The time scale of the decoherence depends on the strength of the infrared contributions of the interaction and on properties of the initial state of the Boson system. These results are first derived for a Hamiltonian with conservation laws. But in the most general case the Hamiltonian includes an additional scattering potential, and the only conserved quantity is the energy of the total system. The superselection sectors remain stable against the perturbation by the scattering processes.

1 Introduction

Superselection rules are the basis for the emergence of classical physics within quantum theory. But despite of the great progress in understanding superselection rules, see e.g. [25], quantum mechanics and quantum field theory do not provide enough exact superselection rules to infer the classical probability of "facts" from quantum theory. This problem is most often discussed in the context of measurement of quantum mechanical objects. In an important paper about the process of measurement Hepp [11] has presented a class of models for which the dynamics induces superselection sectors. Hepp starts with a very large algebra of observables – essentially all observables with the exception of the "observables at infinity" which constitute an a priory set of superselection rules – and the superselection sectors emerge in the weak operator convergence. But it has soon been realized that the algebra of observables, which is relevant for the understanding of the process of measurement [10] [3] and, more generally for the understanding of the classical appearance of the world [26] [13] [14] can be severely restricted. Then strong or even uniform operator convergence is possible.

A system, which is weakly coupled to an environment, which has a Hamiltonian with a continuous spectrum, usually decays into its ground state, if the environment is in a normal state; or the system approaches a canonical ensemble, if the environment is in a state with positive temperature. More interesting decoherence effects may occur on an intermediate time scale, or in systems, for which the decay or the thermalization are prevented by conservation laws. To emphasize effects on an intermediate time scale one can use a strong coupling between system and environment. This method has some similarity to the singular coupling method of the Markov approximation, which also scales the dynamics at an intermediate time period to large times. The basic model, which we discuss, has therefore the following properties: existence of conservation laws and strong coupling. Thereby strong coupling means that the spectral properties of the Hamiltonian are modified by the interaction term.

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In this paper, which is an extension of [16], we investigate the emergence of superselection rules for a system, which is coupled to a mass zero Boson field. The dynamics of the total system is always generated by a semibounded Hamiltonian. The restriction to the Boson sector corresponds to a van Hove model [24]. As the main result we prove for a class of such models:

– The superselection rules are induced by the infrared contributions of the Boson field.

– The superselection sectors are stable for $t \to \infty$ if and only if the Boson field is infrared divergent.

The infrared divergence of the van Hove model has been studied by Schroer [23] more than forty years ago. The Boson field is still defined on the Fock space, but the ground state of the Boson field disappears in the continuum. In the usual discussions of decoherence this type of infrared divergence corresponds to the ohmic or subohmic case [19]. As additional result we prove that the induced superselection sectors are stable against perturbation by scattering processes.

The paper is organized as follows. In Sect. 2 we give a short introduction to the dynamics of subsystems and to superselection rules induced by the environment. The calculations are preformed in the Schrödinger picture, which allows also non-factorized initial states. We prove that the off-diagonal matrix elements of the reduced statistical operator can be suppressed in trace norm for discrete and for continuous superselection rules. In Sect. 3 we investigate a class of Hamiltonian models with the environment given by a mass zero Boson field, and the interplay between infrared divergence and induced superselection rules is derived. The resulting superselection sectors do not depend on the initial state; they finally emerge for all initial states of the total system. But to have superselection sectors, which are effective on a short time scale, the reference state of the environment has to satisfy some "smoothness" conditions.

In Sect. 3.4 we admit a KMS state of positive temperature as reference state of the Boson system. Again the same superselection sectors emerge, even on a shorter time scale. In the final Sect. 4 we prove that the induced superselection sectors are stable against additional scattering processes. Some technical details for the Sects. 2 and 3 are given in the Appendices A and B.

2 Induced superselection rules

2.1 General considerations

We start with a few mathematical notations. Let $\mathcal{H}$ be a separable Hilbert space, then the following spaces of linear operators are used.

$\mathcal{B}(\mathcal{H})$: The linear space of all bounded operators A with the operator norm $\|A\|$.

$\mathcal{T}(\mathcal{H})$: The linear space of all nuclear operators A with the trace norm $\|A\|_1 = \operatorname{tr}\sqrt{A^+A}$.

$\mathcal{S}(\mathcal{H})$: The set of all positive nuclear operators W with a normalized trace, $\operatorname{tr} W = 1$.

If A is a closed (unbounded) linear operator, then $\mathcal{D}(A) \subset \mathcal{H}$ denotes the domain of definition of this operator.

With the exception of Sect. 3.4, where also KMS states are admitted for the environment, we assume standard quantum mechanics where any state of a quantum system is represented by a statistical operator $W \in \mathcal{S}(\mathcal{H})$; the rank one projection operators thereby correspond to the pure states. Without additional knowledge about the structure of the system we have to assume that the set of all states corresponds to $\mathcal{S}(\mathcal{H})$, and the operator algebra of all (bounded) observables coincides with $\mathcal{B}(\mathcal{H})$.

In the following we consider an *open system*, i.e. a system S which interacts with

an environment E, such that the total system $S \times E$ satisfies the usual Hamiltonian dynamics. The Hilbert space $\mathcal{H}_{S\times E}$ of the total system is the tensor space $\mathcal{H}_S \otimes \mathcal{H}_E$ of the Hilbert spaces for S and for E. Let $W \in \mathcal{S}(\mathcal{H}_{S\times E})$ be the state of the total system and $A \in \mathcal{B}(\mathcal{H}_S)$ be an observable of the system S, then the expectation $\mathrm{tr}_{S\times E} W(A \otimes I_E)$ satisfies the identity $\mathrm{tr}_{S\times E}(A \otimes I_E)W = \mathrm{tr}_S A\rho$ with the reduced statistical operator $\rho_S = \mathrm{tr}_E W \in \mathcal{S}(\mathcal{H}_S)$. Here the symbols tr_S, tr_E and $\mathrm{tr}_{S\times E}$ denote the (partial) traces with respect to the Hilbert spaces $\mathcal{H}_S$, $\mathcal{H}_E$ or $\mathcal{H}_{S\times E}$, respectively. We shall refer to $\rho_S = \mathrm{tr}_E W$ as the *state* of the system S. As indicated above we consider the usual Hamiltonian dynamics for the total system, i.e. $W \to W(t) = U(t)WU^+(t) \in \mathcal{S}(\mathcal{H}_{S\times E})$ with the unitary group $U(t) = \exp(-iH_{S\times E}t)$ generated by the total Hamiltonian $H_{S\times E}$. Except for the trivial case that S and E do not interact, the dynamics of the reduced statistical operator $\rho_S(t) = \mathrm{tr}_E\, U(t)WU^+(t)$ does no longer follow a group law; and it is exactly this dynamics which can produce induced superselection sectors.

In order to define a linear dynamics $\rho_S \to \rho_S(t)$ for the state of the system S we have to assume that the initial state factorizes as

$$W = \rho_S \otimes \rho_E, \tag{1}$$

see [18]. Here $\rho_S \in \mathcal{S}(\mathcal{H}_S)$ is the initial state of the system and $\rho_E \in \mathcal{S}(\mathcal{H}_E)$ is the reference state of the environment. The dynamics of the reduced statistical operator $\rho_S(t)$ then follows as

$$\rho_S \in \mathcal{S}(\mathcal{H}_S) \to \rho_S(t) = \Phi_t(\rho_S) := \mathrm{tr}_E U(t)\,(\rho_S \otimes \rho_E)\,U^\dagger(t) \in \mathcal{S}(\mathcal{H}_S). \tag{2}$$

The reduced dynamics $\Phi_t(\rho_S)$ can be extended to a continuous linear mapping $\rho_S \in \mathcal{T}(\mathcal{H}_S) \to \Phi_t(\rho_S) \in \mathcal{T}(\mathcal{H}_S)$ with the obvious properties

$$\|\Phi_t(\rho_S)\|_1 \le \|\rho_S\|_1\,,\ \mathrm{tr}_S\Phi_t(\rho_S) = \mathrm{tr}_S\rho_S,\ \Phi_t(\rho_S) \ge 0 \text{ if } \rho_S \ge 0. \tag{3}$$

Here $\|\cdot\|_1$ is the trace norm of operators on $\mathcal{H}_S$.

Discrete and continuous superselection rules are characterized by a self-adjoint superselection operator $F = \int_{\mathbb{R}} \lambda P(d\lambda)$. The projection operators $P(\Delta)$ of the spectral resolution of this operator are defined for all intervals $\Delta = [a, b)$ of the real line and satisfy

$$\begin{array}{l} P(\Delta^1 \cup \Delta^2) = P(\Delta^1) + P(\Delta^2) \quad \text{if} \quad \Delta^1 \cap \Delta^2 = \emptyset \\ P(\Delta^1)P(\Delta^2) = P(\Delta^1 \cap \Delta^2),\ P(\emptyset) = 0,\ P(\mathbb{R}) = 1. \end{array} \tag{4}$$

The mapping $\Delta \to P(\Delta)$ can be extended to the σ-algebra of the real line $\mathcal{B}(\mathbb{R})$ generated by open sets. In the most commonly discussed case of discrete superselection rules the function $x \in \mathbb{R} \to P((-\infty, x))$ is a step function. In the case of continuous superselection rules, in which we are mainly interested in, the function $x \in \mathbb{R} \to P((-\infty, x))$ is strongly continuous, and the projection operators $P((a, b)) = P([a, b)) = P([a, b])$ coincide.

The dynamics of the total system $S \times E$ induces superselection rules into the system S, if there exists a family of projection operators $\{P_S(\Delta) \mid \Delta \subset \mathbb{R}\}$ on the Hilbert space $\mathcal{H}_S$, which satisfies the rules (4), such that the off-diagonal parts $P_S(\Delta^1)\Phi_t(\rho_S)P_S(\Delta^2)$ of the statistical operators of the system S are dynamically suppressed, i.e.
$P_S(\Delta^1)\Phi_t(\rho_S)P_S(\Delta^2) \to 0$ if $t \to \infty$ and $\mathrm{dist}(\Delta^1, \Delta^2) > 0$. In the subsequent sections we derive superselection rules, for which the off-diagonal parts of the statistical operator even vanish in trace norm

$$\left\|P_S(\Delta^1)\Phi_t(\rho_S)P_S(\Delta^2)\right\|_1 \to 0 \quad \text{if } t \to \infty \tag{5}$$

for all initial states $\rho_S \in \mathcal{S}(\mathcal{H}_S)$ and all separated intervals Δ^1 and Δ^2. This statement, which does not specify the time scale of the decoherence process, can be used as definition of induced superselection rules. But to have superselection rules, which contribute to the emergence of classical properties, the decrease of (5) has to be sufficiently fast. We shall come back to that problem later.

2.2 Models

For all models we are investigating, the total Hamiltonian is defined on the tensor space $\mathcal{H}_{S\times E} = \mathcal{H}_S \otimes \mathcal{H}_E$ as

$$\begin{aligned} H_{S\times E} &= H_S \otimes I_E + I_S \otimes H_E + F \otimes G \\ &= \left(H_S - \frac{1}{2}F^2\right) \otimes I_E + \frac{1}{2}\left(F \otimes I_E + I_S \otimes G\right)^2 + I_S \otimes \left(H_E - \frac{1}{2}G^2\right) \end{aligned} \tag{6}$$

where H_S is the positive Hamiltonian of S, H_E is the positive Hamiltonian of E, and $F \otimes G$ is the interaction potential between S and E with operators F on $\mathcal{H}_S$ and G on $\mathcal{H}_E$. To guarantee that $H_{S\times E}$ is self-adjoint and semibounded we assume

1) The operators F and F^2 (G and G^2) are essentially self-adjoint on the domain of H_S (H_E). The operators $H_S - \frac{1}{2}F^2$ and $H_E - \frac{1}{2}G^2$ are semibounded.

Since $F^2 \otimes I_E \pm 2F \otimes G + I_S \otimes G^2$ are positive operators, the operator $F \otimes G$ is $(H_S \otimes I_E + I_S \otimes H_E)$-bounded with relative bound one, and Wüst's theorem, see e.g. Theorem X.14 in [22], implies that $H_{S\times E}$ is essentially self-adjoint on the domain of $H_S \otimes I_E + I_S \otimes H_E$. Moreover $H_{S\times E}$ is obviously semibounded.

To derive induced superselection rules we need the rather severe restriction

2) The operators H_S and F commute strongly, i.e. their spectral projections commute.

This assumption implies that F is a conserved quantity of the dynamics generated by the Hamiltonian (6). The operator F has a spectral representation

$$F = \int_{\mathbb{R}} \lambda P_S(d\lambda) \tag{7}$$

with a family (4) of projection operators $P_S(\Delta)$ indexed by measurable subsets $\Delta \subset \mathbb{R}$. We shall see below that exactly the projection operators of this spectral representation determine the induced superselection sectors.

As a consequence of assumption 2) we have $[H_S, P_S(\Delta)] = 0$ for all intervals $\Delta \subset \mathbb{R}$. The Hamiltonian (6) has therefore the form $H_{S\times E} = H_S \otimes I_E + \int_{\mathbb{R}} P_S(d\lambda) \otimes (H_E + \lambda G)$. The operator $|G| = \sqrt{G^2}$ has the upper bound $|G| \leq aG^2 + (4a)^{-1}I$ with an arbitrarily small constant $a > 0$. Since G^2 is H_E-bounded with relative bound 2, the operator G is H_E-bounded with an arbitrarily small bound. The Kato-Rellich theorem, see e.g. [22], implies that the operators $H_E + \lambda G$ are self-adjoint on the domain of H_E for all $\lambda \in \mathbb{R}$. The unitary evolution $U(t) := \exp(-iH_{S\times E}t)$ of the total system can therefore be written as
$U(t) = (U_S(t) \otimes I_E) \int dP_S(\lambda) \otimes \exp\left(-i\left(H_E + \lambda G\right)t\right)$, where

$$U_S(t) = \exp\left(-iH_S t\right) \tag{8}$$

is the unitary evolution of the system S. The evolution (2) of an initial state $\rho_S \in \mathcal{S}(\mathcal{H}_S)$ follows as

$$\Phi_t(\rho_S) = U_S(t) \left(\int_{\mathbb{R}\times\mathbb{R}} \chi(\alpha, \beta; t)\, P_S(d\alpha)\, \rho_S\, P_S(d\beta) \right) U_S^+(t) \tag{9}$$

with the trace

$$\chi(\alpha, \beta; t) = \mathrm{tr}_E \left(\mathrm{e}^{i(H_E+\alpha G)t} \mathrm{e}^{-i(H_E+\beta G)t} \rho_E \right). \tag{10}$$

For the models investigated in Sect. 3 this trace factorizes into

$$\chi(\alpha, \beta; t) = \mathrm{e}^{i\vartheta(\alpha,t)} \chi_0(\alpha - \beta; t) \mathrm{e}^{-i\vartheta(\beta,t)} \tag{11}$$

where $\vartheta(\alpha, t)$ is a real phase. The function $\chi_0(\lambda; t) = \overline{\chi_0(-\lambda; t)}$ and its derivative can be estimated by

$$|\chi_0(\lambda; t)| \leq \phi\left(\lambda^2 \zeta(t)\right), \quad \int_\delta^\infty \left| \frac{\partial}{\partial\lambda} \chi_0(\lambda; t) \right| d\lambda \leq \phi\left(\delta^2 \zeta(t)\right) \text{ for all } \delta \geq 0. \tag{12}$$

Thereby $\phi(s)$ is a positive non increasing function which vanishes for $s \to \infty$ such that $\int^\infty \phi(s) ds < \infty$, and the time dependent positive function $\zeta(t)$ increases to infinity if $t \to \infty$. The factorization (10) implies that the operator (9) is the product $\Phi_t(\rho_S) = U_S(t) U_\vartheta(t) \left(\int_{\mathbb{R}\times\mathbb{R}} \chi_1(\alpha - \beta; t)\, P_S(d\alpha)\, \rho_S\, P_S(d\beta) \right) U_\vartheta^+(t) U_S^+(t)$ with the unitary operator $U_\vartheta(t) = \int \exp(i\vartheta(\alpha, t))\, P_S(d\alpha)$. As the projection operators $P_S(\Delta)$ commute with the unitary operators $U_S(t)$ and $U_\vartheta(t)$, the trace norm of $P_S(\Delta^1)\Phi_t(\rho_S)P_S(\Delta^2)$ is given by

$$\left\| P_S(\Delta^1)\Phi_t(\rho_S)P_S(\Delta^2) \right\|_1 = \left\| \int_{\Delta^1\times\Delta^2} \chi_0(\alpha - \beta; t)\, P_S(d\alpha)\, \rho_S\, P_S(d\beta) \right\|_1. \tag{13}$$

The phase function does not contribute to this norm. In Appendix A we prove that the estimate (12) is sufficient to derive the upper bound

$$\left\| P_S(\Delta^1)\Phi_t(\rho_S)P_S(\Delta^2) \right\|_1 \leq \phi\left(\delta^2 \zeta(t)\right) \tag{14}$$

for arbitrary intervals Δ^1 and Δ^2 with a distance $\delta \geq 0$. This estimate is uniform for all initial states $\rho_S \in \mathcal{S}(\mathcal{H}_S)$. The arguments of Appendix A are applicable to superselection operators (7) F with an arbitrary spectrum. For operators $F = \sum \lambda_n P_n^S$ with a discrete spectrum, which has no accumulation point, uniform norm estimates can be derived with simpler methods, see [15] or Sect. 7.6 of [14].

The norm (14) vanishes for all intervals with a positive distance $\mathrm{dist}(\Delta^1, \Delta^2) = \delta > 0$ on a time scale, which depends on the functions $\zeta(t)$ and $\phi(s)$. The function $\zeta(t)$ is mainly determined by the Hamiltonian, whereas $\phi(s)$ depends strongly on the reference state of the environment.

Remark 1 *A simple class of explicitly soluble models, which yield estimates similar to (14), can be obtained under the additional assumptions*
– the operator G has an absolutely continuous spectrum,
– the Hamiltonian H_E and the potential G commute strongly.
Models of this type have been investigated (for operators F with a discrete spectrum) by Araki [3] and by Zurek [26], see also Sect. 7.6 of [14] and [15]. With these additional assumptions the trace (10) simplifies to $\chi(\alpha, \beta; t) = \mathrm{tr}_E\left(\mathrm{e}^{i(\alpha-\beta)Gt}\rho_E\right)$. Let $G = \int_{\mathbb{R}} \lambda P_E(d\lambda)$ be the spectral representation of the operator G. Then the measure $d\mu(\lambda) :=$

$\mathrm{tr}_E\,(P_E(d\lambda)\,\rho_E)$ *is absolutely continuous with respect to the Lebesgue measure for any* $\rho_E \in \mathcal{S}(\mathcal{H}_E)$*, and the function* $\chi(t) := \mathrm{tr}_E\left(\mathrm{e}^{iGt}\rho_E\right) = \int_{\mathbb{R}} \mathrm{e}^{i\lambda t}\; d\mu(\lambda)$ *vanishes for* $t \to \infty$*. Under suitable restrictions on the reference state the measure* $d\mu(\lambda) = \mathrm{tr}_E\,(P_E(d\lambda)\,\rho_E)$ *has a smooth density, and we can derive a fast decrease of the Fourier transform* $\chi(t)$ *and its derivatives. That implies upper bounds similar to (12) and a fast decrease of (10)* $\chi(\alpha, \beta; t)$ *for* $\alpha \neq \beta$*.*

Remark 2 *Instead of the dynamics (2) in the Schrödinger picture we can use the Heisenberg dynamics*

$$A \in \mathcal{B}(\mathcal{H}_S) \to \Psi_t(A) = \Psi_t(A) := \mathrm{tr}_E U^\dagger(t)\,(A \otimes I_E)\,U(t)\rho_E \in \mathcal{B}(\mathcal{H}_S) \tag{15}$$

to investigate induced superselection rules. As the estimate (14) is uniform with respect to the initial state $\rho_S \in \mathcal{S}(\mathcal{H}_S)$*, the duality relation* $\mathrm{tr}_S\left(P_S(\Delta^1)\Phi_t(\rho_S)P_S(\Delta^2)A\right) = \mathrm{tr}_S\,\rho_S\Psi_t\left(P_S(\Delta^2)AP_S(\Delta^1)\right)$ *leads to a criterion for induced superselection rules in the Heisenberg picture:*

$$\lim_{t\to\infty} \left\| \Psi_t\left(P_S(\Delta^1)AP_S(\Delta^2)\right)\right\| = 0 \tag{16}$$

for all observables $A \in \mathcal{B}(\mathcal{H}_S)$ *and for all intervals* Δ^1 *and* Δ^2 *with a distance* $\mathrm{dist}(\Delta^1, \Delta^2) > 0$*. In the case of models with the Hamiltonian (6) which satisfy Assumption 2), the condition (16) is equivalent to a more transparent condition. For these models the full dynamics* $U(t) = \exp(-iH_{S\times E}t)$ *commutes with* $P_S(\Delta) \otimes I_E$*, and the Heisenberg dynamics (15) satisfies the identities* $P_S(\Delta)\Psi_t(A) = \Psi_t(P_S(\Delta)A)$ *and* $\Psi_t(A)P_S(\Delta) = \Psi_t(AP_S(\Delta))$*. These identities and (16) imply that the off-diagonal parts of* $\Psi_t(A)$ *have to vanish for all observables* $A \in \mathcal{B}(\mathcal{H}_S)$ *and for all disjoint intervals with a non-vanishing distance at large* t

$$\lim_{t\to\infty} \left\| P_S(\Delta^1)\Psi_t\,(A)\,P_S(\Delta^2)\right\| = 0. \tag{17}$$

This criterion resembles the definition of the exact superselection rules: $P_S(\Delta^1)\,A\,P_S(\Delta^2) = 0$ *for all* $\Delta^1 \cap \Delta^2 = \emptyset$*, see e.g. [12] or [25]. The criterion (17) has been used in [16] to derive induced superselection rules for the model of Sect. 3.*

3 The interaction with a Boson field

3.1 The Hamiltonian

We choose a system S which satisfies the constraints 1) and 2), and the environment E is given by a Boson field. As specific example we may consider a spin system with Hilbert space $\mathcal{H}_S = \mathbb{C}^2$ and Hamiltonian $H_S = \alpha\sigma_3$ and $F = \beta\sigma_3$ where $\alpha \geq 0$ and β are real constants and σ_3 is the Pauli spin matrix. A more interesting example is a particle on the real line with velocity coupling. The Hilbert space of the particle is $\mathcal{H}_S = \mathcal{L}^2(\mathbb{R})$. The Hamiltonian and the interaction potential of the particle are

$$H_S = \frac{1}{2}P^2 \text{ and } F = P \tag{18}$$

where $P = -i\,d/dx$ is the momentum operator of the particle. The identity $H_S - \frac{1}{2}F^2 = 0$ guarantees the positivity of the first term in (6).

As Hilbert space $\mathcal{H}_E$ we choose the Fock space of symmetric tensors $\mathcal{F}(\mathcal{H}_1)$ based on the one particle Hilbert space $\mathcal{H}_1$. The Hamiltonian is generated by a one-particle Hamilton operator M on $\mathcal{H}_1$ with the following properties

(i) M is a positive operator with an absolutely continuous spectrum,
(ii) M has an unbounded inverse M^{-1}.

The spectrum of M is (a subset of) $\mathbb{R}_+$, which – as a consequence of the second assumption – includes zero. The Hamiltonian of the free field is then the derivation $H_E = d\Gamma(M)$ generated by M, see Appendix B. As explicit example we may take $\mathcal{H}_1 = \mathcal{L}^2(\mathbb{R}^n)$ with inner product $(f \mid g) = \int_{\mathbb{R}^n} \overline{f(k)} g(k) d^n k$. The one-particle Hamilton operator can be chosen as $(Mf)(k) := \varepsilon(k) f(k)$ with the positive energy function $\varepsilon(k) = c\,|k|\,,\, c > 0,\, k \in \mathbb{R}^n$. Let $a_k^{\#}$, $k \in \mathbb{R}^n$, denote the distributional creation/annihilation operators, such that $a^+(f) = \int a_k^+ f(k) d^n k$ and $a(f) = \int a_k \overline{f(k)} d^n k$ are the creation/annihilation operators of the vector $f \in \mathcal{H}_1$, normalized to $[a(f), a^+(g)] = (f \mid g)$. The Hamiltonian $H_E = d\Gamma(M)$ coincides with $H_E = \int \varepsilon(k) a_k^+ a_k d^n k$. The interaction potential G is chosen as the self-adjoint field operator

$$G = \Phi(h) := a^+(h) + a(h) \tag{19}$$

where the vector $h \in \mathcal{H}_1$ satisfies the additional constraint

$$2\left\| M^{-\frac{1}{2}} h \right\| \leq 1. \tag{20}$$

This constraint guarantees that $H_E - \frac{1}{2}\Phi^2(h)$ is bounded from below, and the Hamiltonian (6) is a well defined semibounded operator on $\mathcal{H}_S \otimes \mathcal{F}(\mathcal{H}_1)$, see Appendix B. In the sequel we always assume that (20) is satisfied.

To derive induced superselection sectors we have to estimate the time dependence of the traces (10) $\chi(\alpha, \beta; t) = \mathrm{tr}_E U_{\alpha\beta}(t) \rho_E$ where ρ_E is the reference state of the Boson field, and the unitary operators $U_{\alpha\beta}(t)$ are given by

$$U_{\alpha\beta}(t) := \exp(iH_\alpha t) \exp(-iH_\beta t), \text{ with } H_\alpha = H_E + \alpha\Phi(h),\ \alpha, \beta \in \mathbb{R}. \tag{21}$$

The Hamiltonians H_α are Hamiltonians of the van Hove model [24]. Details for the following statements are given in the Appendix B. The Hamiltonian $H_E + \Phi(h)$ is defined on the Fock space $\mathcal{F}(\mathcal{H}_1)$ as semibounded self-adjoint operator if $h \in \mathcal{H}_1$ is in the domain of $M^{-\frac{1}{2}}$, $h \in \mathcal{D}(M^{-\frac{1}{2}})$. But this Hamiltonian has a ground state only if the low energy contributions of h are not too strong, more precisely, if

$$h \in \mathcal{D}(M^{-1}) \tag{22}$$

is satisfied. Under this more restrictive condition the Hamiltonian has another important property: $H_E + \Phi(h)$ is unitarily equivalent to the free Hamiltonian H_E

$$H_E + \Phi(h) = T^+(M^{-1}h) H_E T(M^{-1}h) - \left\| M^{-\frac{1}{2}} h \right\|^2. \tag{23}$$

Thereby the intertwining operators are the unitary Weyl operators $T(f) = \exp\left(a^+(f) - a(f)\right)$ defined for $f \in \mathcal{H}_1$.

3.2 Coherent states as reference state

For the further calculations we first choose as reference state a coherent state. Let $f \in \mathcal{H}_1 \to \exp f = 1_{vac} + f + \frac{1}{2} f \circ f + .. \in \mathcal{F}(\mathcal{H}_1)$ be the convergent exponential series of the symmetric tensor algebra of the Fock space. Thereby $1_{vac} \in \mathcal{F}(\mathcal{H}_1)$ is the vacuum vector. Then $T(f) 1_{vac} = \exp\left(f - \frac{1}{2}\|f\|^2\right)$ is a normalized exponential vector or coherent

state. The reference state ρ_E is the projection operator $\omega(f)$ onto this vector, i.e. $\omega(f) = T(f)P_{vac}T^{+}(f)$ where P_{vac} is the projection operator onto the vacuum. The basic identity which characterizes the coherent states is the expectation of the Weyl operators

$$\mathrm{tr}_E T(h)\omega(f) = \exp\left(-\frac{1}{2}\|h\|^2\right)\exp\left(2i\,\mathrm{Im}\,(f \mid h)\right) \tag{24}$$

Under the assumption (22) the trace (10) is calculated in Appendix B using (23) and properties of the Weyl operators. The result is

$$\mathrm{tr}_E U_{\alpha\beta}(t)\omega(f) = \exp\left(-(\alpha-\beta)^2\zeta(t)\right)\exp\left(i\left(\vartheta(\alpha,t)-\vartheta(\beta,t)\right)\right) \tag{25}$$

with

$$\zeta(t) = \frac{1}{2}\left\|\left(I-\exp(iMt)\right)M^{-1}h\right\|^2; \tag{26}$$

the phase function $\vartheta(\alpha,t)$ is given in (57). This result implies an estimate (12) of the trace where $\phi(s)$ is the exponential

$$\phi(s) = \exp\left(-s\right) \tag{27}$$

and $\zeta(t)$ is the function (26).

In this first step the identities (25) and (26) have been derived assuming (22). But under this restriction the function (26) is almost periodic. It may grow to large numbers, but it cannot diverge to infinity. Hence the traces (25) do not vanish for $t \to \infty$. One can achieve a strong decrease which persists for some finite time interval; but inevitably, recurrences exist.

To derive induced superselection rules one has to violate the condition (22). If $h \in \mathcal{D}(M^{-\frac{1}{2}}) \setminus \mathcal{D}(M^{-1})$ we prove in Appendix B that the identities (25) and (26) are still valid. Then an evaluation of (26) implies that $\zeta(t)$ diverges for $t \to \infty$, and superselection rules follow from (14). The time scale of the decoherence depends only on the vector h in the interaction potential (19), and (26) can increase like $\log t$ or also like t^{α} with some $\alpha \in (0,1)$, see (62) and (63). The assumption $h \notin \mathcal{D}(M^{-1})$ is therefore necessary and sufficient for the emergence of superselection rules, which persist for $t \to \infty$. Exactly under this condition the Boson field is known to be infrared divergent. It is still defined on the Fock space, but the bare Boson number diverges and its ground state disappears in the continuum, see [23] [2].

3.3 Arbitrary normal states as initial state

The results of Sect. 3.2 can be easily extended to reference states which are superpositions of a finite number of exponential vectors, see Appendix B. Estimates like (14) remain valid with an additional numerical factor, which increases with the number of exponential vectors involved. The linear span $\mathcal{L}\{\exp f \mid f \in \mathcal{H}_1\}$ of the exponential vectors is a dense linear subset of the Fock space $\mathcal{H}_E = \mathcal{F}(\mathcal{H}_1)$, and the convex linear span of all projection operators onto these vectors is a dense subset $\mathcal{S}_{coh} \subset \mathcal{S}(\mathcal{H}_E)$ of all states of the Boson system. We finally obtain for all reference states $\rho_E \in \mathcal{S}_{coh}$ an estimate like (14)

$$\left\|P_S(\Delta^1)\Phi_t(\rho_S)P_S(\Delta^2)\right\|_1 \leq c(\rho_E)\phi\left((1-\varepsilon)\delta^2\zeta(t)\right), \tag{28}$$

where ζ and ϕ are again the functions (26) and (27), but with some small $\varepsilon > 0$ and an additional numerical factor $c(\rho_E)$ which depends on the reference state. If
$h \in \mathcal{D}(M^{-\frac{1}{2}}) \setminus \mathcal{D}(M^{-1})$ this estimate implies for all $\rho_E \in \mathcal{S}_{coh}$

$$\lim_{t\to\infty}\left\|P_S(\Delta^1)\Phi_t(\rho_S)P_S(\Delta^2)\right\|_1 = 0 \tag{29}$$

if the intervals Δ^1 and Δ^2 are separated by a distance $\delta > 0$. Due to the factor $c(\rho_E)$ the emergence of the superselection sectors $\{P_S(\Delta)\mathcal{H}_S\}$ is not uniform with respect to ρ_E; but for suitably restricted subsets of reference states a fast suppression of the off-diagonal matrix elements of $\Phi_t(\rho_S)$ can be achieved.

So far we have assumed that the initial state factorizes. The Schrödinger picture allows to start from the more general initial states

$$W = \sum_{\mu=1}^{N} c_\mu \, \rho_{S\mu} \otimes \rho_{E\mu} \tag{30}$$

with $\rho_{S\mu} \in \mathcal{S}(\mathcal{H}_S),\ \rho_{E\mu} \in \mathcal{S}_{coh}$ and real (positive and negative) numbers c_μ, which satisfy $\sum_\mu c_\mu = \operatorname{tr} W = 1$. Thereby N is an arbitrary finite number. The set of states (30) is dense in $\mathcal{S}(\mathcal{H}_{S+E})$ and will be denoted by $\mathcal{S}_{fin}(\mathcal{H}_{S+E})$. The reduced dynamics for such an initial state

$$\rho_S(t) = \widehat{\Phi}_t(W) := \operatorname{tr}_E U(t) W U^\dagger(t) \tag{31}$$

decomposes into $\rho_S(t) = \sum_\mu c_\mu \Phi_t^\mu(\rho_{S\mu})$, where $\Phi_t^\mu(\,.\,)$ is the reduced dynamics (2) with reference state $\rho_{E\mu}$. For all contributions $\Phi_t^\mu(\rho_{S\mu})$ estimates of the type (28) are valid. Hence $\left\| P_S(\Delta^1)\widehat{\Phi}_t(W)P_S(\Delta^2)\right\|_1 \leq \sum_\mu |c_\mu| \left\| P_S(\Delta^1)\Phi_t^\mu(\rho_{S\mu})P_S(\Delta^2)\right\|_1$ implies

$$\lim_{t\to\infty} \left\| P_S(\Delta^1)\widehat{\Phi}_t(W)P_S(\Delta^2)\right\|_1 = 0 \tag{32}$$

for $W \in \mathcal{S}_{fin}(\mathcal{H}_{S+E})$ and all separated intervals.

By a continuity argument on the mapping (31) we finally derive that the superselection sectors $\{P_S(\Delta)\mathcal{H}_S \mid \Delta \subset \mathbb{R}\}$ emerge for all initial states $W \in \mathcal{S}(\mathcal{H}_{S+E})$ of the total system.

Theorem 1 *If the interaction is determined by a vector $h \in \mathcal{D}(M^{-\frac{1}{2}})\backslash\mathcal{D}(M^{-1})$ with norm restriction (20), then (32) is true for all initial states $W \in \mathcal{S}(\mathcal{H}_{S+E})$ and all intervals with distance* $\operatorname{dist}\left(\Delta^1, \Delta^2\right) > 0$.

Proof. The mapping (31) can be extended to a linear mapping $W \in \mathcal{T}(\mathcal{H}_{S+E}) \to \widehat{\Phi}_t(W) \in \mathcal{T}(\mathcal{H}_S)$ which is continuous with respect to the trace norms of these spaces

$$\left\|\widehat{\Phi}_t(W)\right\|_1 \leq \|W\|_1 \,. \tag{33}$$

Given some $\varepsilon > 0$ and a state $W \in \mathcal{S}(\mathcal{H}_{S+E})$, then we can find a $W_1 \in \mathcal{S}_{fin}(\mathcal{H}_{S+E})$ such that $\|W - W_1\|_1 < \varepsilon$. The limit (32) implies: for intervals Δ^1 and Δ^2 with a non-vanishing distance there is a time $T(\varepsilon) < \infty$ such that $\left\| P_S(\Delta^1)\widehat{\Phi}_t(W_1)P_S(\Delta^2)\right\|_1 < \varepsilon$ if $t > T(\varepsilon)$. By linearity of $\widehat{\Phi}_t$ we have $\widehat{\Phi}_t(W) = \widehat{\Phi}_t(W_1) + \widehat{\Phi}_t(W - W_1)$, and we derive the upper bound

$$\begin{aligned}
&\left\| P_S(\Delta^1)\widehat{\Phi}_t(W)P_S(\Delta^2)\right\|_1 \\
&\leq \left\| P_S(\Delta^1)\widehat{\Phi}_t(W_1)P_S(\Delta^2)\right\|_1 + \left\| P_S(\Delta^1)\widehat{\Phi}_t(W - W_1)P_S(\Delta^2)\right\|_1 \\
&\leq \left\| P_S(\Delta^1)\widehat{\Phi}_t(W_1)P_S(\Delta^2)\right\|_1 + \|W - W_1\|_1 < 2\varepsilon
\end{aligned} \tag{34}$$

As ε can be chosen arbitrarily small, the Theorem follows. ■

3.4 KMS states as reference states

The considerations presented so far can be extended to an environment with positive temperature $\beta^{-1} > 0$. That means the Boson field is in a KMS state[2], which is uniquely characterized by the following expectation of the Weyl operators $T(h)$

$$\langle T(h) \rangle_\beta = \exp\left(- \left(h \mid \left((\mathrm{e}^{\beta M} - I)^{-1} + \frac{1}{2} \right) h \right) \right). \tag{35}$$

The calculations for a KMS state correspond to the calculations for coherent states. Only the expectation (24) has to be substituted by the expectation (35). As $(\exp(\beta M) - I)^{-1}$ is a positive operator we have

$$\langle T(h) \rangle_\beta < \exp\left(-\frac{1}{2} \|h\|^2 \right) = \left| \mathrm{tr}_E T(h) \omega(f) \right|. \tag{36}$$

Hence in an environment of temperature $\beta^{-1} > 0$ the superselection sectors are induced on shorter time scale than for coherent states, see Appendix B.

4 Scattering processes

In this final section we investigate the stability of the induced superselection sectors against additional scattering processes. We restrict the initial state of the total system to a normal state $W \in \mathcal{S}(\mathcal{H}_{S+E})$ to apply standard scattering theory.

The Hamiltonian (6) is generalized to

$$H = H_{S\times E} + V, \tag{37}$$

where V is a scattering potential on $\mathcal{H}_{S\times E} = \mathcal{H}_S \otimes \mathcal{H}_E$. There are no constraints on the commutators $[H_{S\times E}, V]$ or $[F \otimes I_E, V]$, and in general the dynamics has no conservation law except energy conservation. The restriction to scattering potentials means that the wave operator $\Omega = \lim_{t\to\infty} U^+(t) U_0(t)$ with $U(t) = \exp(-itH)$ and $U_0(t) = \exp(-itH_{S\times E})$ exists as strong limit. To simplify the arguments we assume that there are no bound states and that the wave operator is unitary on $\mathcal{H}_{S\times E}$. Then the time evolution $U(t) = \exp(-itH)$ behaves asymptotically like $U_0(t)\Omega^+$ with $U_0(t) = \exp(-itH_{S\times E})$. More precisely, the existence of wave operators implies

$$\lim_{t\to\infty} \left\| U(t) W U^+(t) - U_0(t) \Omega^+ W \Omega U_0^+(t) \right\|_1 = 0 \tag{38}$$

for all $W \in \mathcal{S}(\mathcal{H}_{S+E})$. As Ω is unitary we have $\Omega^+ W \Omega \in \mathcal{S}(\mathcal{H}_{S\times E})$ for all $W \in \mathcal{S}(\mathcal{H}_{S+E})$. Let us denote the reduced dynamics with the full Hamiltonian (37) by

$$\rho_S(t) = \widehat{\Phi}_t(W) = \mathrm{tr}_E \left(U(t) W U^+(t) \right), \tag{39}$$

and the reduced dynamics with the Hamiltonian (6) by $\widehat{\Phi}_t^0(W) = \mathrm{tr}_E \left(U_0(t) W U_0^+(t) \right)$. The linearity of the trace implies

$$\widehat{\Phi}_t(W) = \widehat{\Phi}_t^0(\Omega^+ W \Omega) + \mathrm{tr}_E \left(U(t) W U^+(t) - U_0(t) \Omega^+ W \Omega U_0^+(t) \right). \tag{40}$$

[2]The KMS states of an environment which has a Hamiltonian with a continuous spectrum cannot be represented by a statistical operator in $\mathcal{S}(\mathcal{H}_E)$. In such a case the algebra of observables has to be restricted to the Weyl algebra, which is strictly smaller than $\mathcal{B}(\mathcal{H}_E)$, and the KMS states are positive linear functionals on that algebra.

The off-diagonal contributions of the reduced dynamics (39) with scattering can therefore be estimated by

$$\begin{aligned} \left\| P_S(\Delta^1)\rho_S(t)P_S(\Delta^2)\right\|_1 \;\; &\leq \;\; \left\| P_S(\Delta^1)\ \widehat{\Phi}_t^0(\Omega^+ W\Omega)\ P_S(\Delta^2)\right\|_1 \\ &\quad + \left\| U(t)WU^+(t) - U_0(t)\Omega^+ W\Omega U_0^+(t)\right\|_1 . \end{aligned} \quad (41)$$

As $\Omega^+ W\Omega \in \mathcal{S}(\mathcal{H}_{S\times E})$ the first term vanishes for $t \to \infty$ under the conditions of Theorem 1, and the second term vanishes for $t \to \infty$ as a consequence of (38). Hence we have derived

Theorem 2 *If the interaction is determined by a vector* $h \in \mathcal{D}(M^{-\frac{1}{2}}) \setminus \mathcal{D}(M^{-1})$ *with norm restriction (20), then*

$$\lim_{t\to\infty} \left\| P_S(\Delta^1)\rho_S(t)P_S(\Delta^2)\right\|_1 = 0 \quad (42)$$

follows for the reduced dynamics (39) with the Hamiltonian (37) for all initial states $W \in \mathcal{S}(\mathcal{H}_{S+E})$ *and all intervals with distance* $\operatorname{dist}\left(\Delta^1, \Delta^2\right) > 0$.

Therefore scattering processes do not destroy or modify the induced superselection sectors $\{P_S(\Delta)\mathcal{H}_{S.} \mid \Delta \subset \mathbb{R}\}$, but the time scale of there emergence increases. Estimates on the time scale require a more detailed investigation of the scattering process, which is not given here.

5 Conclusion

We have investigated a class of systems, which are coupled to a mass zero Boson field. These models exhibit the following properties:

- The Boson field induces superselection rules into the system, if and only if the field is infrared divergent. Thereby infrared divergence means that the bare Boson number diverges and the Boson vacuum disappears in the continuum, but the Hamiltonian remains bounded from below.
- The superselection sectors are fully determined by the Hamiltonian, they finally emerge for all normal initial states of the total system, – including non-product states – and for KMS states as reference states of the Boson system.
- The time scale of the decoherence depends on the interaction and on the initial state. There are restrictions on the reference state of the Boson field to obtain superselection rules, which are effective within a short time.
- The superselection sectors persist, if additional scattering processes take place. In this case the total system may have no conservation law except energy conservation.

These results underline the known importance of low frequency excitations of the environment for the process of decoherence [19] [8].

A Estimates of operators

Let $P : \Delta = [a, b) \subset \mathbb{R} \to \mathcal{L}(\mathcal{H})$ be a family of orthogonal projectors in $\mathcal{H}$ with the properties (4). The mapping $P(\Delta)$ can be extended to a σ-additive measure on the Borel algebra $\mathcal{B}(\mathbb{R})$ generated by open subsets of the real line $\mathbb{R}$. The operators $P([a, b))$ are naturally left continuous in both variables a and b. In what follows we investigate some integrals of bounded operator-valued functions with respect to P. More details can be found in [17].

Lemma 1 *Let $f : \mathbb{R} \to \mathcal{T}(\mathcal{H})$ be a differentiable function with a Bochner integrable derivative $f'(x) \in \mathcal{T}(\mathcal{H})$. Then for any interval $\Delta = [a, b) \subset \mathbb{R}$ the following identity holds*

$$\begin{aligned} \int_a^b P(dx)f(x) &= P([a,b))f(b) - \int_a^b P([a,x))f'(x)dx \\ &= P([a,b))f(a) + \int_a^b P([x,b))f'(x)dx, \end{aligned} \tag{43}$$

and the norm of this integral has the upper bound

$$\left\| \int_\Delta P(dx)f(x) \right\| \leq \min\left(\|f(a)\|, \|f(b)\|\right) + \int_\Delta \left\|f'(x)\right\|_1 dx. \tag{44}$$

Proof. The identities (43) are just the integration by parts formula of the Stieltjes integral, see e.g. [4] Sect. 5.1. The norm estimate (44) is then a consequence of $\|P(\Delta)\| \leq 1$ and the rule $\|AB\|_1 \leq \|A\| \, \|B\|_1$ for the trace norm. ■

The same type of identities and norm estimates can be derived for integrals $\int_\Delta f(x)P(dx) = \left(\int_\Delta P(dx)f^+(x)\right)^+$ with a reversed order of the operators. An immediate consequence of Lemma 1 is the

Corollary 2 *Let $f : \mathbb{R} \to \mathcal{T}(\mathcal{H})$ be a function with a Bochner integrable derivative $f'(x) \in \mathcal{T}(\mathcal{H})$. If $\|f(x)\|_1$ vanishes for $x \to \pm\infty$ the identities*

$$\begin{aligned} \int_a^\infty P(dx)f(x) &= -\int_a^\infty P([a,x))f'(x)dx \quad \text{and} \\ \int_{-\infty}^b P(dx)f(x) &= \int_{-\infty}^b P([x,b))f'(x)dx \end{aligned} \tag{45}$$

hold for all $a, b \in \mathbb{R}$ and the estimate

$$\left\| \int_\Delta P(dx)f(x) \right\|_1 \leq \int_\Delta \left\|f'(x)\right\|_1 dx. \tag{46}$$

follows for the infinite intervals $\Delta = [a, \infty)$ and $(-\infty, b)$.

We now consider operators

$$S_\varphi = \int_{\mathbb{R}\times\mathbb{R}} \varphi(x,y)P(dx)SP(dy) \tag{47}$$

where $S \in \mathcal{T}(\mathcal{H})$ and $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is a differentiable function. We obviously have $P(\Delta^1)S_\varphi P(\Delta^2) = \int_{\Delta^1\times\Delta^2} \varphi(x,y)P(dx)SP(dy)$. First let us notice that

$$P(\Delta^1)S_\varphi P(\Delta^2) = \int_{\Delta^1} P(dx)S \int_{\Delta^2} \varphi(x,y)P(dy) = \int_{\Delta^1} P(dx)SA(x), \tag{48}$$

where the function $A(x)$ is defined by $A(x) = \int_{\Delta^2} \varphi(x,y)P(dy) \in \mathcal{L}(H)$. Its derivative is $A'(x) = \int_{\Delta^2} \varphi_1(x,y)P(dy)$ with $\varphi_1(x,y) = \frac{\partial}{\partial x}\varphi(x,y)$. The operator norm of this derivative has the upper bound $\|A'(x)\| \leq \sup_{y\in\Delta^2} |\varphi_1(x,y)|$. Then (48) can be estimated by Corollary 2. We formulate the final result for operators (46) S_φ with a function $\varphi(x,y) = \chi(x-y)$ which depends only on the difference $x - y$.

Theorem 3 *Let $\chi : x \in \mathbb{R} \to \mathbb{C}$ be a differentiable complex-valued function with $\chi(x) \to 0$ if $|x| \to \infty$ and $|\chi'(x)| \leq \phi(|x|)$, where $\phi(s)$ is non-increasing for $s \geq 0$ with a bounded integral $\int_0^\infty \phi(x)dx < \infty$. Then for any nuclear operator S the operator S_φ with $\varphi(x,y) = \chi(x-y)$ is again nuclear, and for the disjoint intervals $\Delta^1 = (-\infty, b_1)$ and $\Delta^2 = [a_2, \infty)$ with $\delta = a_2 - b_1 \geq 0$ the following estimate holds*

$$\left\| P(\Delta^1) S_\varphi P(\Delta^2) \right\|_1 \leq \|S\|_1 \int_\delta^\infty \phi(x)dx. \tag{49}$$

B The van Hove model

B.1 The Hamiltonian

Let $F \circ G$ denote the symmetric tensor product of the Fock space $\mathcal{F}(\mathcal{H}_1)$ with vacuum 1_{vac}. For all $f \in \mathcal{H}_1$ the exponential vectors $\exp f = 1_{vac} + f + \frac{1}{2} f \circ f + ...$ converge within $\mathcal{F}(\mathcal{H}_1)$, the inner product being $(\exp f \mid \exp g) = \exp(f \mid g)$. Coherent vectors (states) are the normalized exponential vectors $\exp\left(f - \frac{1}{2}\|f\|^2\right)$. The linear span of all exponential vectors $\{\exp f \mid f \in \mathcal{H}_1\}$ is dense in $\mathcal{F}(\mathcal{H}_1)$. The creation operators $a^+(f)$ are uniquely determined by $a^+(f) \exp g = f \circ \exp g =$
$\frac{\partial}{\partial \lambda} \exp(f + \lambda g) \mid_{\lambda=0}$ with $f, g \in \mathcal{H}_1$ and the annihilation operators are given by $a(g) \exp f = (g \mid f) \exp f$. These operators satisfy the standard commutation relations $[a(f), a^+(g)] = (f \mid g)$. If M is a operator on $\mathcal{H}_1$ then $\Gamma(M)$ is uniquely defined as operator on $\mathcal{F}(\mathcal{H}_1)$ by $\Gamma(M) \exp f := \exp(Mf)$, and the derivation $d\Gamma(M)$ is defined by $d\Gamma(M) \exp f := (Mf) \circ \exp f$.

For arbitrary elements $g \in \mathcal{H}_1$ the unitary Weyl operators are defined on the set of exponential vectors by $T(g) \exp f = \exp\left(-(g \mid f) - \frac{1}{2}\|g\|^2\right) \exp(f+g)$. This definition is equivalent to $T(g) = \exp(a^+(g) - a(g))$. The Weyl operators are characterized by the properties

$$\begin{aligned} T(g_1)T(g_2) = T(g_1+g_2) \exp(-i \operatorname{Im}(g_1 \mid g_2)) \\ (1_{vac} \mid T(g)\, 1_{vac}) = \exp\left(-\tfrac{1}{2}\|g\|^2\right). \end{aligned} \tag{50}$$

The matrix element of $T(h)$ between coherent vectors $\exp\left(f - \frac{1}{2}\|f\|^2\right) = T(f)1_{vac}$ follows from these relations as

$$\left(1_{vac} \mid T^+(g)T(h)T(f)\, 1_{vac}\right) = \exp\left(-\frac{1}{2}\|h+f-g\|^2 + i \operatorname{Im}\{(g \mid f) + (f+g \mid h)\}\right). \tag{51}$$

For a free field the time evolution on the Fock space is given by $U(t) = \exp(-iH_E t) = \Gamma(V(t))$ with $V(t) := \exp(-iMt)$. For exponential vectors we obtain $U(t) \exp f = \exp(V(t)f)$. From these equations the dynamics of the Weyl operators follows as

$$U^+(t)T(g)U(t) = T\left(V^+(t)\, g\right). \tag{52}$$

For fixed $h \in \mathcal{H}_1$ the unitary operators $T^+(h)U(t)T(h)$, $t \in \mathbb{R}$, form a one parameter group which acts on exponential vectors as
$T^+(h)U(t)T(h) \exp f = \exp\left((h \mid V(t)(f+h) - f) - \|h\|^2\right) \exp(V(t)(f+h) - h)$.
For $h \in \mathcal{H}_1$ with $Mh \in \mathcal{H}_1$ the generator of this group is easily identified with
$T^+(h)H_E T(h) = H_E + \Phi(Mh) + (h \mid Mh)$, where $\Phi(.)$ is the field operator. This identity

was first derived by Cook [6] by quite different methods. If h satisfies $M^{-1}h \in \mathcal{H}_1$ we obtain

$$T^{+}(M^{-1}h)H_E T(M^{-1}h) - \left\| M^{-\frac{1}{2}}h \right\|^2 = H_E + \Phi(h) \tag{53}$$

which is the Hamiltonian of the van Hove model [24], see also, [5] p.166ff, [9] and [2].

For all $h \in \mathcal{H}_E$ with $M^{-\frac{1}{2}}h \in \mathcal{H}_E$ the field operator $\Phi(h)$ satisfies the estimate

$$\|\Phi(h)\psi\| \leq 2 \left\| M^{-\frac{1}{2}}h \right\| \left\| \sqrt{H_E}\psi \right\| + \|h\| \|\psi\|, \tag{54}$$

where $\psi \in \mathcal{F}(\mathcal{H}_1)$ is an arbitrary vector in the domain of H_E, see e.g. eq. (2.3) of [1]. As consequences we obtain the following Lemma for the Hamiltonian of the van Hove model, see [23] and [2].

Lemma 3 *The operators $H_E + \lambda\Phi(h)$, $\lambda \in \mathbb{R}$, are self-adjoint on the domain of H_E if $h \in \mathcal{D}(M^{-\frac{1}{2}})$.*

Proof. From (54) and the numerical inequality $\sqrt{x} \leq ax + (4a)^{-1}$, valid for $x \geq 0$ and $a > 0$, we obtain a bound $\|\Phi(h)\psi\| \leq c_1 \|H_E\psi\| + c_2 \|\psi\|$ with positive numbers c_1, $c_2 > 0$ where c_1 can be chosen arbitrarily small. Then the Kato-Rellich Theorem yields the first statement. ■

A further consequence is

Lemma 4 *The operator $H_E - \frac{1}{2}\Phi^2(h)$ has the lower bound $H_E - \frac{1}{2}\Phi^2(h) \geq -\|h\|^2$, if $h \in \mathcal{H}_1$ and $\left\| M^{-\frac{1}{2}}h \right\| \leq 2^{-1}$.*

Proof. From (54) we obtain
$\|\Phi(h)\psi\|^2 \leq 4 \left\| M^{-\frac{1}{2}}h \right\|^2 (\psi \mid H_E\psi) + 4 \left\| M^{-\frac{1}{2}}h \right\| \|h\| \left\| \sqrt{H_E}\psi \right\| \|\psi\| + \|h\|^2 \|\psi\|^2$
$\leq 8 \left\| M^{-\frac{1}{2}}h \right\|^2 (\psi \mid H_E\psi) + 2 \|h\|^2 \|\psi\|^2$. Hence the operator inequalities
$0 \leq \frac{1}{2}\Phi^2(h) \leq 4 \left\| M^{-\frac{1}{2}}h \right\|^2 H_E + \|h\|^2 I_E$ hold, and Lemma 4 follows. ■

Therefore the total Hamiltonian (6) is semibounded, and the unitary operators $U_\lambda(t) = \exp(-i(H_E + \lambda\Phi(h))t)$ are well defined if (20) is satisfied.

B.2 Evaluation of the traces

In a first step we evaluate the expectation value of (21) $U_{\alpha\beta}(t) = U_\alpha(-t)U_\beta(t)$ for a coherent state (= normalized exponential vector) $\exp\left(f - \frac{1}{2}\|f\|^2\right) = T(f)\, 1_{vac}$ under the additional constraint $h \in \mathcal{D}(M^{-1})$. This assumption allows to use the identity (53) which reduces all calculations to the Weyl relations and the vacuum expectation (51). The extension to the general case, which violates $h \in \mathcal{D}(M^{-1})$, can then be performed by a continuity argument.

If $M^{-1}h \in \mathcal{H}_1$ the identity (53) implies
$U_\lambda(t) = T(-\lambda M^{-1}h)U_0(t)T(\lambda M^{-1}h) \exp\left(i\lambda^2 \left(h \mid M^{-1}h\right) t\right)$. Then $U_{\alpha\beta}(t) = U_\alpha(-t)U_\beta(t)$ can be calculated with the help of (50) and (52) as

$$\begin{aligned} U_{\alpha\beta}(t) &= T\left((\alpha - \beta)\left(V^{+}(t) - I\right) M^{-1}h\right) \exp\left(-i\eta(t)\right), \\ \eta(t) &= -(\alpha^2 - \beta^2)\left\{\left(h \mid M^{-1}h\right) t - \left(M^{-1}h \mid M^{-1}\sin(Mt)h\right)\right\}. \end{aligned} \tag{55}$$

The matrix element of $U_{\alpha\beta}(t)$ between the coherent states $T(f)\,1_{vac}$ and $T(g)\,1_{vac}$ is then evaluated with the help of (51)

$$\begin{aligned}(1_{vac} \mid T^{+}(g)U_{\alpha\beta}(t)T(f)\,1_{vac}) = \exp\left(-\tfrac{1}{2}\left\|(\alpha-\beta)\left(V^{+}(t)-I\right)M^{-1}h+f-g\right\|^{2}\right) \\ \times\exp\left(i\,\mathrm{Im}(g\mid f)\right)\times\exp\left(i\left(\vartheta(\alpha,t)-\vartheta(\beta,t)\right)\right)\end{aligned} \tag{56}$$

with the phase function

$$\vartheta(\alpha,t) = -\alpha\,\mathrm{Im}\left(f+g \mid \left(I-V^{+}(t)\right)M^{-1}h\right) - \alpha^{2}\left(M^{-1}h \mid ht - M^{-1}\sin(Mt)h\right). \tag{57}$$

For $g=f$ the identity (56) leads to the trace (25).

So far we have assumed $h\in\mathcal{D}(M^{-1})$. Then the norm $\left\|(V^{+}(t)-I)\,M^{-1}h\right\| = \left\|(I-\exp(iMt))\,M^{-1}h\right\|$ is an almost periodic function of t, and induced superselection rule can emerge only in an approximate sense on an intermediate time scale. But $(V^{+}(t)-I)\,M^{-1}h$ is a vector in $\mathcal{H}_1$ also under the weaker condition $h\in\mathcal{D}(M^{-\frac{1}{2}})\supset\mathcal{D}(M^{-1})$. Moreover from Lemma 3 we know that the van Hove Hamiltonians $H_E+\lambda\Phi(h)$ and the groups $U_\lambda(t)$ are defined under this weaker condition. In the next step we shall use a continuity argument to prove that (56) is indeed still valid for vectors $h\in\mathcal{D}(M^{-\frac{1}{2}})$ without knowing whether $h\in\mathcal{D}(M^{-1})$ or not. Then we derive the essential statement that the norm $\left\|(V^{+}(t)-I)\,M^{-1}h\right\|$ diverges for $t\to\infty$ if $h\notin\mathcal{D}(M^{-1})$. As this behaviour is possible under the condition (20), which guarantees the existence of a semibounded Hamiltonian (6), stable superselection sectors emerge if h is chosen such that $h\in\mathcal{D}(M^{-\frac{1}{2}})\setminus\mathcal{D}(M^{-1})$ with the additional constraint (20).

For the proof of this statement we introduce the norm

$$|||h||| := \|h\| + \left\|M^{-\frac{1}{2}}h\right\|. \tag{58}$$

Let $h_n\in\mathcal{H}_1$, $n=1,2,...$, be a sequence of real vectors which converges in this topology to a vector h, then we know from (54) and the proof of Lemma 3 that there exist two null sequences of positive numbers c_{1n} and c_{2n} such that

$$\|(\Phi(h_n)-\Phi(h))\,\psi\| \le c_{1n}\,\|(H_E+\Phi(h))\,\psi\| + c_{2n}\,\|\psi\|.$$

Hence the operators $H_E+\Phi(h_n)$ converge strongly to $H_E+\Phi(h)$ and the groups $U(h_n;t) = \exp\left(-i\left(H_E+\Phi(h_n)\right)t\right)$ converge strongly to the group $U(h;t)=\exp\left(-i\left(H_E+\Phi(h)\right)t\right)$, uniformly in any finite interval $0\le t\le s<\infty$; see e.g. Theorem 4.4 on p. 82 of [20], or Theorem 3.17 of [7]. The operators
$U_{\alpha\beta,n}(t) := \exp\left(i\left(H_E+\alpha\Phi(h_n)\right)t\right)\exp\left(-i\left(H_E+\beta\Phi(h_n)\right)t\right)$ converge therefore in the weak operator topology to $U_{\alpha\beta}(t)$. For $n=1,2,..$ we can calculate the corresponding traces $\mathrm{tr}_E U_{\alpha\beta,n}(t)\omega(f)$ with the result (26), where h has to be substituted by h_n. Since (26) is continuous in the variable h in the topology (58) the limit for $n\to\infty$ is again given by (26).

To prove the divergence of $\left\|(V^{+}(t)-I)\,M^{-1}h\right\|$ for $t\to\infty$ we introduce the spectral resolution $P_M(d\lambda)$ of the one-particle Hamilton operator M. The energy distribution of the vector $h\in\mathcal{H}_1$ is given by the measure $d\sigma_h(\lambda)=(h\mid P_M(d\lambda)h)$. The exponent (26) is then the integral

$$\zeta(t) = \frac{1}{2}\left\|(I-\exp(iMt))\,M^{-1}h\right\|^{2} = 2\int_{\mathbb{R}_+}\lambda^{-2}\sin^{2}\frac{\lambda t}{2}\,d\sigma_h(\lambda). \tag{59}$$

This integral is well defined for all $h \in \mathcal{H}_1$, and $\zeta(t)$ is a differentiable function for $t \in \mathbb{R}$. The requirement $h \in \mathcal{D}(M^{-\frac{1}{2}}) \backslash \mathcal{D}(M^{-1})$ is equivalent to the conditions $\int_0^\infty \lambda^{-1} d\sigma_h(\lambda) < \infty$ and

$$\int_\varepsilon^\infty \lambda^{-2}\, d\sigma_h(\lambda) \nearrow \infty \quad \text{if } \varepsilon \to +0. \tag{60}$$

Lemma 5 *If $h \notin \mathcal{D}(M^{-1})$, i.e. (60), the integral (59) diverges for $t \to \infty$.*

Proof. Since the operator M has an absolutely continuous spectrum, the measure $d\sigma_h(\lambda)$ is absolutely continuous with respect to the Lebesgue measure $d\lambda$ on $\mathbb{R}_+$. Consequently, the measure $\lambda^{-2}\, d\sigma_h(\lambda)$ is absolutely continuous with respect to the Lebesgue measure on any interval (ε, ∞) with $\varepsilon > 0$. The identity $\sin^2 \frac{\lambda t}{2} = \frac{1}{2}(1 - \cos \lambda t)$ and the Lebesgue Lemma therefore imply
$\lim_{t\to\infty} \int_\varepsilon^\infty \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda) = \frac{1}{2} \int_\varepsilon^\infty \lambda^{-2}\, d\sigma_h(\lambda)$. Given a number $\Lambda > 0$ the assumption (60) yields the existence of an $\varepsilon > 0$ such that

$$\lim_{t\to\infty} \int_\varepsilon^\infty \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda) = \frac{1}{2} \int_\varepsilon^\infty \lambda^{-2}\, d\sigma_h(\lambda) > \Lambda. \tag{61}$$

From the inequality $\int_{\mathbb{R}_+} \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda) \geq \int_\varepsilon^\infty \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda)$ we then obtain $\int_0^\infty \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda) > \Lambda$ for sufficiently large t. Since the number Λ can be arbitrarily large the integral (59) diverges for $t \to \infty$. ∎

If $d\sigma_h(\lambda)$ satisfies additional regularity conditions, we can obtain more precise statements. A powerlike behaviour $d\sigma_h(\lambda) \cong c \cdot \lambda^{2\mu} d\lambda$, $c > 0$, near $\lambda = +0$ is compatible with the requirement $h \in \mathcal{D}(M^{-\frac{1}{2}}) \setminus \mathcal{D}(M^{-1})$ if $0 < \mu \leq \frac{1}{2}$. For the ohmic case $d\sigma_h(\lambda) \cong c \cdot \lambda d\lambda$ we obtain

$$\begin{aligned} \zeta(t) &= 2 \int_0^\infty \lambda^{-2} \sin^2 \frac{\lambda t}{2}\, d\sigma_h(\lambda) = \int_0^\infty \lambda^{-2} (1 - \cos \lambda t)\, d\sigma_h(\lambda) \\ &\simeq c \int_0^t s^{-1} (1 - \cos s)\, ds \simeq c \log t \quad \text{for } t \to \infty; \end{aligned} \tag{62}$$

and the subohmic case $d\sigma_h(\lambda) \cong c \cdot \lambda^{2\mu} d\lambda$ with $0 < \mu < \frac{1}{2}$ implies a powerlike divergence

$$\begin{aligned} \zeta(t) &= \int_0^\infty \lambda^{-2} (1 - \cos \lambda t)\, d\sigma_h(\lambda) \\ &\simeq c\, t^{1-2\mu} \int_0^t s^{-2+2\mu} (1 - \cos s)\, ds \sim t^{1-2\mu} \quad \text{for } t \to \infty. \end{aligned} \tag{63}$$

So far the reference state ω has been a coherent state. But the results remain true if we take as reference state ω the projection onto a vector $\psi = \sum_{n=1}^N c_n \exp f_n$, $f_n \in \mathcal{H}_1$, which is a finite linear combination of exponential vectors. In that case the trace (10) is a sum of terms (56) with f and g given by the vectors f_n. The exponent $\zeta_{\alpha\beta}[f,g](t) := \frac{1}{2} \left\| (\alpha - \beta)(V^+(t) - I) M^{-1} h + f - g \right\|^2$ in (56) diverges for $\alpha \neq \beta$ under the same condition as (59) does. Moreover, the asymptotic behaviour of $\zeta_{\alpha\beta}[f,g](t)$ is dominated by the asymptotics of $(\alpha - \beta)^2 \zeta(t)$. Hence uniform estimates like (14) remain valid, but the function $\phi\left(\delta^2 \zeta(t)\right)$ should be substituted by $c \cdot \phi\left((1-\varepsilon)\delta^2 \zeta(t)\right)$ with a small $\varepsilon > 0$, and a constant $c \geq 1$, which depends on the coefficients c_n and on the norms $\|f_m - f_n\|$. This factor increases with the number N of the exponential vectors.

In the case of a KMS state of temperature $\beta^{-1} > 0$ the calculations essentially follow the calculations for coherent states. The expectation of $U_{\mu\nu}(t)$ is calculated using (35). The result

$$\langle U_{\mu\nu}(t)\rangle_\beta = \exp\left(-(\mu-\nu)^2 \zeta_\beta(t)\right) \exp\left(i\left(\vartheta(\mu,t) - \vartheta(\nu,t)\right)\right) \tag{64}$$

has the same structure as (25) with the temperature dependent function

$$\begin{aligned} \zeta_\beta(t) &= \left(\left(I - \mathrm{e}^{Mt}\right) M^{-1} h \mid \left((\mathrm{e}^{\beta M} - I)^{-1} + \frac{1}{2}\right)\left(I - \mathrm{e}^{Mt}\right) M^{-1} h\right) \\ &\geq \frac{1}{2}\left\|\left(I - \exp\left(Mt\right)\right) M^{-1} h\right\|^2 = \zeta(t), \end{aligned} \tag{65}$$

and the phase function $\vartheta(\mu,t) = -\mu^2\left(M^{-1}h \mid ht + M^{-1}\sin(Mt)h\right)$, which originates from (55). The inequality (65) implies that for $h \in \mathcal{D}(M^{-\frac{1}{2}}) \setminus \mathcal{D}(M^{-1})$ superselection sectors are induced on a shorter time scale than for coherent states.

As a final remark we indicate a modification of the model, which does not use the absolute continuity of the spectrum of M. But we still need a dominating low energy contribution in the interaction. More precisely, we assume that $\sigma_h(\lambda) \equiv \int_0^\lambda d\sigma_h(\alpha)$ behaves at low energies like

$$\lambda^{-2}\sigma_h(\lambda) \nearrow \infty \text{ if } \lambda \to +0. \tag{66}$$

Then we can derive the divergence of (59) by the inequalities $\zeta(t) \geq 4\int_{\mathbf{0}}^{\frac{\pi}{t}} \lambda^{-2}\sin^2\frac{\lambda t}{2}\, d\sigma_h(\lambda) \geq \frac{4}{\pi^2}t^2\int_{\mathbf{0}}^{\frac{\pi}{t}} d\sigma_h(\lambda) = \frac{4}{\pi^2}t^2\,\sigma_h(\frac{\pi}{t})$ using $\sin x \geq \frac{2}{\pi}x$ if $0 \leq x \leq \frac{\pi}{2}$. For measures $d\sigma_h(\lambda) \sim \lambda^{2\mu}d\lambda$ the assumption (66) is more restrictive than (60) – it excludes $d\sigma_h(\lambda) \sim \lambda d\lambda$ which satisfies the conditions of Lemma 2. But (66) is also meaningful for point measures $d\sigma_h(\lambda)$, and M may be an operator with a pure point spectrum. The Boson field can therefore be substituted by an infinite family of harmonic oscillators, which have zero as accumulation point of their frequencies. Such an example has been discussed – also for KMS states – by Primas [21].

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